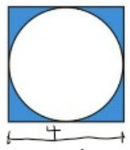
Week 9 2023

Monday, March 20, 2023 9:11 PM



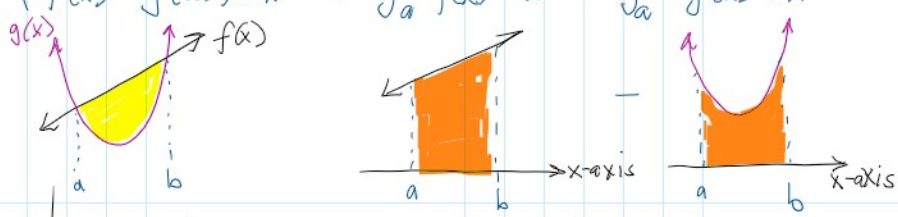
Do: find area of shaded region (BLUE) given circle's radius is 2cm.

$$\begin{aligned}\text{Area}_{\text{shaded region}} &= A_{\square} - A_{\circ} \\ &= bh - \pi r^2 \\ &= 4(4) - \pi \cdot 2^2 \\ &= (16 - 4\pi)\ \text{cm}^2\end{aligned}$$

Section 6.1 More About Areas

Area of a Region Bound (Enclosed) by Multiple Graphs

$$\int_a^b (f(x) - g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$



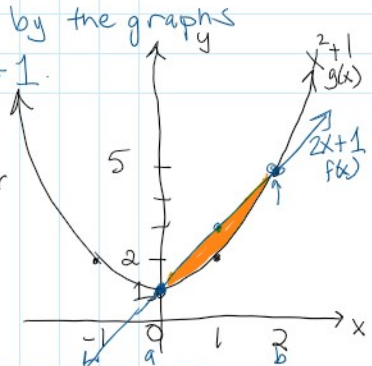
step 1: determine order of subtraction (which curve has larger y-values?)

step 2: determine integration bounds (a, b) from curves' intersections

ex: find the area of the region bound by the graphs $f(x) = 2x + 1$ and $g(x) = x^2 + 1$.

find intersection points - set functions equal to each other

$$\begin{aligned}2x+1 &= x^2+1 \\ 0 &= x^2 - 2x + \cancel{1} - \cancel{1} \\ 0 &= x(x-2) \\ x &= 0 \quad x = 2\end{aligned}$$



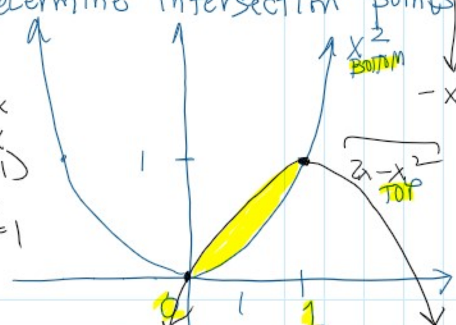
$$\int_0^2 (\underset{\text{line}}{f(x)} - \underset{\text{parabola}}{g(x)})\,dx$$

$$\boxed{\int_a^b x^n\,dx = \frac{x^{n+1}}{n+1}\Big|_a^b}$$

$$\begin{aligned}&= \int_0^2 (2x+1 - (x^2+1))\,dx \\ &= \int_0^2 (2x + \cancel{1} - x^2 - \cancel{1})\,dx \\ &= \left(\frac{2x^2}{2} - \frac{x^3}{3}\right)\Big|_0^2 = (2^2 - 0) - \frac{1}{3}(2^3 - 0) \\ &= 4 \cdot \frac{3}{3} - \frac{8}{3} = \frac{12-8}{3} = \boxed{\frac{4}{3}}\end{aligned}$$

Do: sketch $f(x) = x^2$ and $g(x) = 2x - x^2$ and determine intersection points $(f(x) = g(x))$

$$\begin{aligned}2x - x^2 &= x^2 \\ 0 &= 2x^2 - 2x \\ 0 &= x^2 - x \\ 0 &= x(x-1) \\ x &= 0 \quad x = 1\end{aligned}$$



$-x^2 + 2x + 0$

$$x = \frac{-b}{2a} = \frac{-2}{2(-1)} = 1$$

$$g(1) = 2(1) - 1^2 = 2 - 1 = 1$$

vertex: $(1, 1)$

Do: find area of enclosed region

$$\begin{aligned}\int_0^1 (g(x) - f(x))\,dx &= \int_0^1 (2x - x^2 - x^2)\,dx \\ &= \int_0^1 (2x - 2x^2)\,dx \\ &= 2\int_0^1 (x - x^2)\,dx \\ &= 2\left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 \\ &= 2\left(\frac{1}{2}(1) - \frac{1}{3}(1)\right) \\ &= 2\left(\frac{1}{2}\cdot\frac{3}{3} - \frac{1}{3}\cdot\frac{2}{2}\right) \\ &= 2\left(\frac{3-2}{6}\right) = 2 \cdot \frac{1}{6} = \boxed{\frac{1}{3}}\end{aligned}$$

$$\int_a^b (y_{TOP} - y_{BOTTOM})\,dx$$

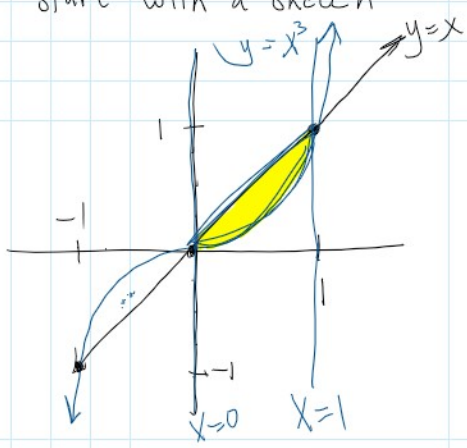
$$= \int_a^b (y_T - y_B)\,dx$$

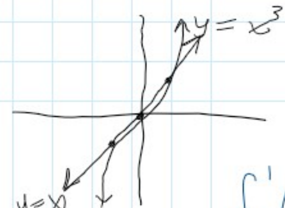
ex. use an integral to find area of a region bounded above by $y = e^x$, bounded below by $y = x$ and ex. use an integral to find area of a region bounded above by $y=e^x$, bounded below by $y=x$ and bounded on sides by vertical lines $x=0$ and $x=1$.

$$\int_0^1 (y_T - y_B)\,dx$$
$$= \int_0^1 (e^x - x)\,dx$$
$$= \left[e^x - \frac{x^2}{2}\right]_0^1$$
$$= e^1 - e^0 - \frac{1}{2}(1-0)$$
$$= e - 1 - \frac{1}{2} = \boxed{e - \frac{3}{2}}$$

Do: find the area of the region bounded by $y=x$, $y=x^3$, $\boxed{x=0,\ x=1}$

start with a sketch





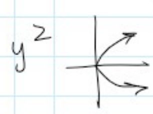
$$\int_0^1 (x - x^3)\,dx$$
$$\frac{x^2}{2} - \frac{x^4}{4}$$

$x = x^3$
$x^3 - x = 0$
$x(x^2-1) = 0$
$x(x+1)(x-1) = 0 \quad x = -1, 1, 0$

$= \boxed{\frac{1}{4}}$

ex. find area enclosed by $y = x-1$ and $y^2 = 2x+6$

instead solve for $\underline{x}$



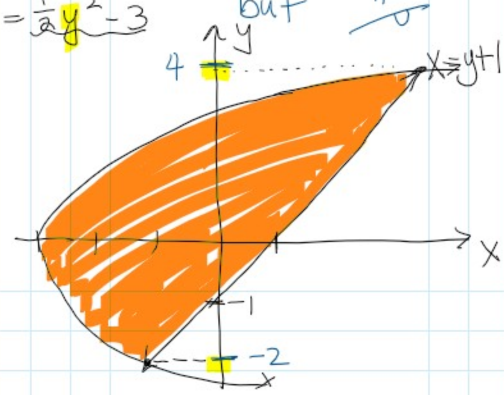
could... $y = \pm\sqrt{2x+6}$

but $\cancel{\text{"}}$

$y = x-1 \longrightarrow x = y+1$

$y^2 = 2x+6$
$\frac{1}{2}(y^2-6) = x = \frac{1}{2}y^2 - 3$

$y+1 = \frac{1}{2}y^2 - 3$
$0 = \frac{1}{2}y^2 - y - 3 - 1$
$2\left[0 = \frac{1}{2}y^2 - y - 4\right]2$
$0 = y^2 - 2y - 8$
$0 = (y-4)(y+2)$
$y=4 \quad y=-2$



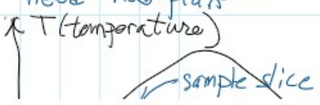
$$\int_{-2}^{4}\left(y+1 - \left(\frac{1}{2}y^2 - 3\right)\right)\underline{\underline{dy}} \longleftarrow \int_{a=y}^{b=y} (x_R - x_L)\,\underline{\underline{dy}}$$
$$= \int_{-2}^{4}\left(y + 1 - \frac{1}{2}y^2 + 3\right)dy$$
$$= \int_{-2}^{4}\left(y + 4 - \frac{1}{2}y^2\right)dy$$
$$= \left[\frac{y^2}{2} + 4y + \frac{1}{2}\cdot\frac{y^3}{3}\right]_{-2}^{4} = F(b) - F(a) = \boxed{18}$$

Section 6.5 Average Value / Mean Value Theorem

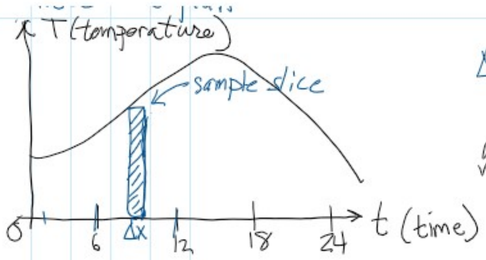
with a finite amount of numbers, to find the average:

$\Rightarrow y_{avg} = \dfrac{y_1 + y_2 + \dots + y_n}{n}$ where n is the number of values being averaged

for infinitely many y values need new plan eg, continuous temperature monitoring



$\Delta x = \dfrac{b-a}{n} \xrightarrow{\text{solve for } n} \boxed{n = \dfrac{b-a}{\Delta x}}$



$\Delta x = \frac{b-a}{n}$ $\xrightarrow{\text{solve for } n}$ $\boxed{n = \frac{b-a}{\Delta x}}$

avg value $= \dfrac{f(x_1^*) + f(x_2^*) + \dots + f(x_n^*)}{\cancel{\frac{b-a}{\Delta x}}} \cdot \dfrac{\Delta x}{b-a}$

$= \frac{1}{b-a}\left[f(x_1^*) + \dots + f(x_n^*)\right]\Delta x$

$= \frac{1}{b-a} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$

$= \frac{1}{b-a} \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$

$$\boxed{f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx}$$

ex. find average value of $f(x) = 1 + x^2$ on $[-1, 2]$ (a, b)

$f_{avg} = \frac{1}{2-(-1)} \int_{-1}^{2} (1+x^2)\,dx$

$= \frac{1}{3} \left(x + \frac{x^3}{3}\right)\Big|_{-1}^{2}$

$= \frac{1}{3}\left(2-(-1) + \frac{1}{3}\left(8 - (-1)^3\right)\right)$ ($(-1)^3 = -1$)

$= \frac{1}{3}\left(3 + \frac{1}{3}\cdot 9\right)$

$= \frac{1}{3}(3+3) = \frac{6}{3} = \boxed{2}$

Do: find average value of $f(x) = \cos^4 x \sin x$ on $[0, \pi]$

$f_{avg} = \frac{1}{\pi}\int_0^{\pi} \cos^4 x \underline{\sin x\,dx}$

$u = \cos x$

$-du = \sin x\,dx$

$x = 0: u = \cos 0 = 1$

$x = \pi: u = \cos\pi = -1$

$= -\frac{1}{\pi}\int_1^{-1} u^4\,du$

$= -\frac{1}{5\pi} u^5 \Big|_1^{-1}$

$= -\frac{1}{5\pi}\left((-1)^5 - 1^5\right)$

$= -\frac{1}{5\pi}(-1-1) = -\frac{1}{5\pi}(-2) = \frac{2}{5\pi}$

$\boxed{f_{avg} = \frac{2}{5\pi}}$

Mean Value Theorem for Integrals

If f is continuous on $[a,b]$ then there exists a number c in $[a,b]$ such that:

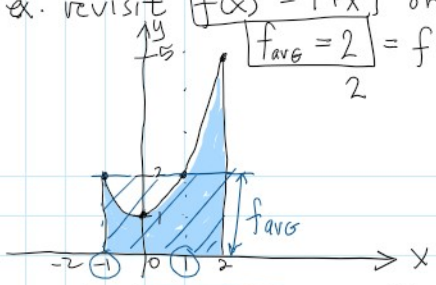
$\rightarrow \boxed{f(c) = f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx}$

ex. revisit $\boxed{f(x) = 1+x^2}$ on $[-1, 2]$

$\boxed{f_{avg} = 2} = f(c) = 1 + c^2$ find c

$2 = 1 + c^2$

$\sqrt{1} = \sqrt{c^2} \Rightarrow c = -1, 1$



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ex. given $\boxed{f(x) = \ln x}$ on $[1, 3]$

a. find average value $\frac{1}{3-1}\int_1^3 \ln x\,dx$

IBP

$u = \ln x \quad dv = dx$

$du = \frac{1}{x}dx \quad v = x$

$\int u\,dv = uv - \int v\,du$

$= \frac{1}{2}\left(x\ln x\Big|_1^3 - \int_1^3 x \cdot \frac{1}{x}\,dx\right)$

$\int 1\,dx$

$= \frac{1}{2}\left(x\ln x\Big|_1^3 - x\Big|_1^3\right)$

$= \frac{1}{2}\left(3\ln 3 - \ln 1 - (3-1)\right)$

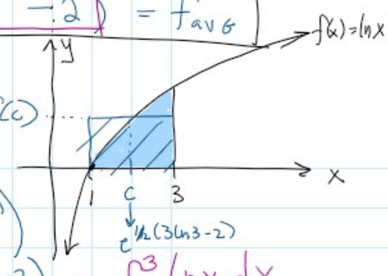
b. find c such that $f(c) = f_{avg}$

$= \boxed{\frac{1}{2}(3\ln 3 - 2) = f_{avg}}$

$\ln c = \frac{1}{2}(3\ln 3 - 2)$

$\boxed{c = e^{1/2(3\ln 3 - 2)}}$

c. show area under curve is same as area of the rectangle



$bh = 2\,f(c) = 2\ln e^{1/2(3\ln 3-2)}$

$= 2 \cdot \frac{1}{2}(3\ln 3 - 2) = \int_1^3 \ln x\,dx$